(b) Since $F = U - \tau\sigma$, $\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V$

Indeed, $dF = dU - \tau d\sigma - \sigma d\tau = \cancel{\tau d\sigma} - pdV - \cancel{\tau d\sigma} - \sigma d\tau$

$$(dF)_V = -\sigma(d\tau)_V \Rightarrow \sigma = -\left(\frac{\partial F}{\partial \tau}\right)_V$$

So $\sigma = \frac{d}{d\tau}\left[\tau \ln\left(1+e^{-\varepsilon/\tau}\right)\right] = \ln\left(1+e^{-\varepsilon/\tau}\right) + \frac{\tau \frac{d}{d\tau}\left(e^{-\varepsilon/\tau}\right)}{1+e^{-\varepsilon/\tau}}$

But $\tau \frac{d}{d\tau}\left(e^{-\varepsilon/\tau}\right) = \tau e^{-\varepsilon/\tau} \frac{\varepsilon}{\tau^2} = \frac{\varepsilon}{\tau} e^{-\varepsilon/\tau}$

Hence, $\sigma = \ln\left(1+e^{-\varepsilon/\tau}\right) + \frac{\varepsilon/\tau\, e^{-\varepsilon/\tau}}{1+e^{-\varepsilon/\tau}}$

Then $U = F + \tau\sigma = -\tau \ln\left(1+e^{-\varepsilon/\tau}\right) + \tau \ln\left(1+e^{-\varepsilon/\tau}\right) + \frac{\varepsilon e^{-\varepsilon/\tau}}{1+e^{-\varepsilon/\tau}}$

$$= \frac{\varepsilon e^{-\varepsilon/\tau}}{1+e^{-\varepsilon/\tau}} = \frac{\varepsilon}{e^{\varepsilon/\tau}+1}$$

$$U = \frac{\varepsilon}{1+e^{\varepsilon/\tau}}$$

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(3) Kittel 3.3

$Z = e^{-0/\tau} + e^{-\hbar\omega/\tau} + \ldots + e^{-s\hbar\omega/\tau} + \ldots = 1 + q + q^2 + \ldots$, where $q = e^{-\hbar\omega/\tau} < 1$

By the formula for the sum of a geometric series (applicable if $|q| < 1$),

$$Z = \frac{1}{1-q} = \frac{1}{1-e^{-\hbar\omega/\tau}}$$

So $F = -\tau \ln Z = -\tau \ln\left(1-e^{-\hbar\omega/\tau}\right)^{-1} = \tau \ln\left(1-e^{-\hbar\omega/\tau}\right)$

$e^x - 1 \approx x$ for small x. So for high τ,