Physics 112 Solutions to H/w #4 Fall 2000

(1) Kittel 3.7, zipper problem

(a) We know that $Z = \sum_s e^{-\varepsilon_s/\tau}$. It's clear that there are only $N+1$ available states: the ground state - with all the links closed, the first excited state - with only one link open (energy ε), and so on. The kth state has k links open and has energy

(1) $$\varepsilon_k = k\varepsilon \quad , k = 0, 1, \ldots, N \quad (k - \#\text{ of open links})$$

Thus $$Z = \sum_{k=0}^{N} e^{-k\varepsilon/\tau} = \sum_{k=0}^{\infty} e^{-k\varepsilon/\tau} - \sum_{k=N+1}^{\infty} e^{-k\varepsilon/\tau} =$$

$$= \sum_{k=0}^{\infty} e^{-k\varepsilon/\tau} - e^{-(N+1)\varepsilon/\tau} \sum_{k=0}^{\infty} e^{-k\varepsilon/\tau} = \sum_{k=0}^{\infty} q^k \left(1 - e^{-(N+1)\varepsilon/\tau}\right),$$

where $q = e^{-\varepsilon/\tau}$. But $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$, so

$$Z = \frac{1 - e^{-(N+1)\varepsilon/\tau}}{1 - e^{-\varepsilon/\tau}}$$

(b) Define $x = e^{-\varepsilon/\tau}$. If $\varepsilon \gg \tau$, $x \to 0$. Also, $Z = \frac{1 - x^{N+1}}{1 - x}$

Now, $P_s = \frac{e^{-\varepsilon_s/\tau}}{Z}$ and $\langle A \rangle = \sum_s A(s) P_s$ (by definition)

s in this sum labels the number of open links for our case. More precisely, since in general s labels the quantum state and in our case ~~each~~ distinct quantum states correspond to distinct numbers of open links, the number of open links is an appropriate label for the quantum states. To find the average of any quantity A, we must first express it as a function of the number of open links. What if A is the identity function, i.e. $A(s) = s$? Then $A(s)$ of course is the number of open links and the average number of open links is

$$\langle A \rangle = \sum_s A(s) P_s = \sum_s s P_s = \sum_s s \frac{e^{-\varepsilon_s/\tau}}{Z} \overset{(1)}{=} \sum_{s=0}^{N} s e^{-s\varepsilon/\tau} \frac{1}{Z} =$$