We now follow the steps leading to formula (3.65)

The 1-particle partition function is

$$Z_1 = \sum_s e^{-\varepsilon_s/\tau} = \sum_{n_x=1}^{\infty} \sum_{n_y=1}^{\infty} \sum_{n_z=1}^{\infty} e^{-\frac{hc}{2L\tau}\sqrt{n_x^2+n_y^2+n_z^2}} \approx$$

$$\approx \int_0^{\infty} dn_x \int_0^{\infty} dn_y \int_0^{\infty} dn_z \; e^{-\frac{hc}{2L\tau}\sqrt{n_x^2+n_y^2+n_z^2}} =$$

$$= \frac{1}{8} \iiint_{\text{all space}} dn_x\, dn_y\, dn_z \; e^{-\frac{hc}{2L\tau}\sqrt{n_x^2+n_y^2+n_z^2}} =$$

$$= \frac{1}{8} \int e^{-\frac{hc}{2L\tau} r}\, dV = \frac{1}{8} \int e^{-\alpha r}\, r^2\, dr\, d\Omega =$$

$$= \frac{4\pi}{8} \int_0^{\infty} r^2 e^{-\alpha r}\, dr$$

Define $\quad I(\alpha) = \int_0^{\infty} e^{-\alpha r}\, dr = \frac{1}{-\alpha} e^{-\alpha r}\Big|_0^{\infty} = \frac{1}{\alpha}$

Then $\quad \frac{dI}{d\alpha} = \int_0^{\infty} \frac{d}{d\alpha}\left(e^{-\alpha r}\right) dr = \int_0^{\infty} -r\, e^{-\alpha r}\, dr$

$$\frac{d^2 I}{d\alpha^2} = \frac{d}{d\alpha}\left(\frac{dI}{d\alpha}\right) = \frac{d}{d\alpha} \int_0^{\infty} -r e^{-\alpha r}\, dr = \int_0^{\infty} \frac{\partial}{\partial \alpha}\left(-r e^{-\alpha r}\right) dr =$$

$$= \int_0^{\infty} r^2 e^{-\alpha r}\, dr$$

But $\quad \frac{d^2 I}{d\alpha^2} = \frac{d}{d\alpha}\left(\frac{-1}{\alpha^2}\right) = \frac{2}{\alpha^3} \left( = \int_0^{\infty} r^2 e^{-\alpha r}\, dr \right)$

So

$$Z_1 = \frac{4\pi}{8} \frac{2}{\alpha^3} = \frac{\pi}{\alpha^3} = \frac{\pi}{(hc/2L\tau)^3} = \text{const}\; \tau^3$$

$$U = \tau^2 \frac{\partial \ln Z_1}{\partial \tau} = \tau^2 \frac{\partial}{\partial \tau}\left(\ln \text{const} + 3 \ln \tau\right) = \tau^2\left(0 + 3\frac{1}{\tau}\right) =$$

$$= 3\tau = U \; !$$